CRYSTALLIZATIONS OF PL 4-MANIFOLDS

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Categories for \( n \)-dimensional manifolds
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**TOP category**

- **topological manifolds**, up to *homeomorphisms*
Categories for $n$-dimensional manifolds

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- topological manifolds, up to homeomorphisms

**PL category**
- triangulated manifolds (PL-manifolds), up to PL-isomorphisms
Categories for \( n \)-dimensional manifolds

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### PL category
- **triangulated manifolds (PL-manifolds)**, up to PL-isomorphisms

### DIFF category
- **smooth manifolds**, up to diffeomorphisms
Classification in dimension 3 and 4

$n=3$

- **TOP=PL** (any topological 3-manifold admits a PL-structure which is unique up to PL-isomorphisms)
- **PL=DIFF** (each PL-structure on a 3-manifold is smoothable in a unique way up to diffeomorphisms)
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- **TOP≠DIFF**
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  - there are topological 4-manifolds admitting no smooth structure;
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- **PL=DIFF** (each PL-structure on a 4-manifold is smoothable and each PL-isomorphism is isotopic to a diffeomorphism)
- **TOP\neq DIFF**
  - there are topological 4-manifolds admitting no smooth structure;
  - there can be non-diffeomorphic smooth structures on the same topological 4-manifold.
4-dimensional classical results

For **closed simply-connected oriented 4-manifolds** the well-known topological classification by Freedman is based on the **intersection form** defined on the second integral cohomology group of the manifold (modulo its torsion).

\[ \text{[Freedman, 1982]} \]

For an even form $\lambda$ there is exactly one homeomorphism class of simply connected closed manifolds having $\lambda$ as intersection form; for an odd form $\lambda$ there are exactly two classes (distinguished by the Kirby-Siebenmann invariant in $\mathbb{Z}_2$), at most one of which admits smooth representatives (smoothness requires vanishing invariant).
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Closed simply-connected smooth 4-manifolds have intersection forms of the following types:

\[ r[1] \oplus r'[-1] \quad s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pm 2nE_8 \oplus t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ with } t > 2n. \]
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[Donaldson, 1983] [Furuta, 2001]

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Up to now there is no classification of smooth structures on any given smoothable topological 4-manifold.
4-dimensional results: TOP vs DIFF

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4-dimensional results: TOP vs DIFF

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Some recent results:

[Akhmedov-Doug Park, 2010], [Akhmedov-Ishida-Doug Park, 2013]

There exist (infinitely many) non-diffeomorphic smooth structures on:

- $\#_{2h-1} \mathbb{CP}^2 \#_{2h} (-\mathbb{CP}^2)$, for any integer $h \geq 1$
- $\#_{2h-1} (S^2 \times S^2)$, for $h \geq 138$
- $\#_{2h-1} (\mathbb{CP}^2 \# (-\mathbb{CP}^2))$, for $h \geq 23$
- $\#_{2p} (S^2 \times S^2)$ and $\#_{2p} (\mathbb{CP}^2 \# (-\mathbb{CP}^2))$, for large enough integers $p$ not divisible by 4.
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- \(\#_{2h - 1}(\mathbb{CP}^2 \#(-\mathbb{CP}^2))\), for \(h \geq 23\)
- \(\#_{2p}(S^2 \times S^2)\) and \(\#_{2p}(\mathbb{CP}^2 \#(-\mathbb{CP}^2))\), for large enough integers \(p\) not divisible by 4.

The existence of exotic PL-structures on \(S^4\), \(\mathbb{CP}^2\), \(S^2 \times S^2\) or \(\mathbb{CP}^2 \#\mathbb{CP}^2\) or \(\mathbb{CP}^2 \#(-\mathbb{CP}^2)\) is still an open problem!
crystallization theory, for $n \geq 4$
In dimension $n \geq 4$, where $TOP \not\cong PL$, a purely combinatorial approach to general PL-manifolds is useful if it yields:

- **combinatorial moves** which realize PL-homeomorphism (and not only TOP-homeomorphism);
- **PL invariants** (possibly distinguishing different PL structures on the same TOP-manifold), whose computation can be performed directly on the combinatorial objects.
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In particular:

- the **gem-complexity** $k(M^n)$ of a PL $n$-manifold $M^n$ is the integer $p - 1$, where $2p$ is the minimum order of a crystallization of $M^n$;
- the **regular genus** $G(M^n)$ of an orientable (resp. non-orientable) PL $n$-manifold $M^n$ is defined as the minimum genus (resp. half the minimum genus) of a surface into which a crystallization of $M^n$ regularly embeds.
[Basak - Casali, 2015]

\[ k(M^4) \geq 3\chi(M^4) + 10rk(M^4) - 6 \]

\[ G(M^4) \geq 2\chi(M^4) + 5rk(M^4) - 4 \]

where:
\[ \chi(M^4) = \text{Euler characteristic of } M^4 \text{ (closed PL 4-manifold)} \]
\[ rk(M^4) = \text{rank of the fundamental group } \pi_1(M^4) \text{ of } M^4 \]
Basic notions on 4-manifolds

PL 4-manifolds via crystallization theory

Classifying PL 4-manifolds by regular genus

Simple and semi-simple crystallizations

Basic elements

Bounds for \(k(M^4)\) and \(G(M^4)\)

TOP classification in the simply-connected case

Regular genus and gem-complexity: lower bounds for \(n = 4\)

[Basak - Casali, 2015]

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where:

\(\chi(M^4)\) = Euler characteristic of \(M^4\) (closed PL 4-manifold)

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In the **simply-connected** case:

\[
k(M^4) \geq 3\beta_2(M^4)
\]

\[
G(M^4) \geq 2\beta_2(M^4)
\]

where \(\beta_2(M^4)\) = second Betti number of \(M^4\)
TOP classification according to regular genus and gem-complexity for \( n = 4 \)
TOP classification according to regular genus and

gem-complexity for $n = 4$

Hence, as a consequence of the up-to-date results about topological classification of simply connected PL 4-manifolds:

[Casali, 2012] [Casali - Cristofori, 2015]

Let $M^4$ be a simply-connected closed PL 4-manifold. If either $k(M^4) \leq 65$ or $G(M^4) \leq 43$, then $M^4$ is TOP-homeomorphic to

$$\left(\#_r\mathbb{C}P^2\right)\#\left(\#_{r'}(-\mathbb{C}P^2)\right) \text{ or } \#_s(S^2 \times S^2),$$

where $r + r' = \beta_2(M^4)$, $s = \frac{1}{2}\beta_2(M^4)$
TOP classification according to regular genus and gem-complexity for \( n = 4 \)

**Sketch of the proof:**
By [Donaldson, 1983], only forms of type
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\begin{array}{c}
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1 & 0
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On the other hand, by [Furuta, 2001], the forms of type

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occur only if \( k > 2n \), i.e. \( \beta_2 \geq 22 \).
TOP classification according to regular genus and gem-complexity for $n = 4$

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The thesis easily follows from the fact that both $k(M^4) \leq 65$ and $G(M^4) \leq 43$ imply $\beta_2 < 22$. 

General properties of the regular genus

For every $n \geq 2$, $G(M^n) = 0 \iff M^n \simeq = \begin{cases} S^n \text{ if } \partial M^n = \emptyset \\ \# D^n \text{ if } \partial M^n \text{ has } h \text{ connected components} \end{cases}$

For every $n$-manifold $M^n (n \geq 3)$, $G(M^n)$ is a non-negative integer invariant, so that $G(M^n) \geq G(\partial M^n)$ and $G(M^n) \geq \text{rk}(M^n)$ where $\text{rk}(M^n)$ denotes the rank of the fundamental group $\pi_1(M^n)$; $G(M^n_1 \# M^n_2) \leq G(M^n_1) + G(M^n_2)$, for any $M^n$.

Conjecture

$G(M^n_1 \# M^n_2) = G(M^n_1) + G(M^n_2)$, for any $M^n_1$, $M^n_2$ closed (orientable) PL $n$-manifolds.
General properties of the regular genus

- For every $n \geq 2$,

$$G(M^n) = 0 \iff M^n \cong \begin{cases} S^n & \text{if } \partial M^n = \emptyset \\ \#_h D^n & \text{if } \partial M^n \text{ has } h \text{ connected components} \end{cases}$$
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  where $\text{rk}(M^n)$ denotes the rank of the fundamental group $\pi_1(M^n)$;

- $G(M_1^n \# M_2^n) \leq G(M_1^n) + G(M_2^n)$, for any $n$.

Conjecture $I_n$

$G(M_1^n \# M_2^n) = G(M_1^n) + G(M_2^n)$, for any $M_1^n$, $M_2^n$ closed (orientable) PL $n$-manifolds.
The case of “low” regular genus

Let $M^4$ be an orientable 4-manifold, with $\partial M^4 = \emptyset$; then:

\[ G(M^4) = \rho \leq 3 \implies M^4 \cong \begin{cases} \#_{\rho}(S^3 \times S^1) \\ \#_{\rho-2}(S^3 \times S^1) \# \mathbb{CP}^2 \end{cases} \]

Let $M^4$ be a non-orientable 4-manifold, with $\partial M^4 = \emptyset$; then:

\[ G(M^4) = \rho \leq 2 \implies M^4 \cong \#_{\rho}(S^3 \tilde{\times} S^1) \]
The case of “low” regular genus

[Casali-Malagoli, 1997]

Let $M^4$ be an (orientable or non-orientable) 4-manifold, with $\partial M^4 \neq \emptyset$; then:

$$G(M^4) = \rho \leq 2 \implies M^4 \cong \begin{cases} \#_{\rho - \partial \rho}(S^3 \tilde{\times} S^1) \# (h)^{\tilde{\bigcirc}_\partial} & \text{if} \ h \geq 1 \\ \mathbb{CP}^2 \# (\#_h D^4) \end{cases}$$

where $0 \leq \partial \rho = G(\partial M^4) \leq \rho$, $h \geq 1$ is the number of boundary components and $(h)^{\tilde{\bigcirc}_\partial}$ denotes the connected sum of $h \geq 1$ orientable or non-orientable 4-dimensional handlebodies of genus $\alpha_i \geq 0$ ($i = 1, \ldots, h$), so that $\sum_{i=1}^h \alpha_i = r$. 

The case of “restricted gap” between regular genus and boundary regular genus

[Casali 1992]

Let $M^4$ be an (orientable or non-orientable) 4-manifold, with $h \geq 1$ boundary components. If $0 \leq m \leq 1$, then:

$$G(M^4) - G(\partial M^4) = m \implies M^4 \cong \#_m (S^3 \times S^1) \# (h) \Y_{\partial \rho}^4.$$
The case of “restricted gap” between regular genus and rank of the fundamental group

[Casali, 1996] [Casali-Malagoli, 1997]

Let $M^4$ be an (orientable or non-orientable) 4-manifold.

- $G(M^4) = rk(M^4) = \rho \iff M^4 \cong \begin{cases} \#_{\rho}(S^3 \tilde{\times} S^1) \\ \#_{\rho-\partial}(S^3 \tilde{\times} S^1) \# (h)^4 \end{cases}$

- $G(M^4) \neq rk(M^4) \implies G(M^4) - rk(M^4) \geq 2$

- $G(M^4) - rk(M^4) = 2$ and $\pi_1(M^4) = *_m\mathbb{Z} \iff M^4 \cong \begin{cases} \#_{\partial}(S^3 \tilde{\times} S^1) \# \mathbb{C}P^2 \\ \#_{\partial}(S^3 \tilde{\times} S^1) \# \mathbb{C}P^2 \# (h)^4 \end{cases}$

- No $M^4$ exists with $\partial M^4 = \emptyset$, $G(M^4) - rk(M^4) = 3$ and $\pi_1(M^4) = *_m\mathbb{Z}$.
Handle-decomposition of PL 4-manifolds

Every closed PL 4-manifold $M^4$ admits a **handle-decomposition**

$$M^4 = H^{(0)} \cup (H_1^{(1)} \cup \cdots \cup H_{r_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{r_2}^{(2)}) \cup (H_1^{(3)} \cup \cdots \cup H_{r_3}^{(3)}) \cup H^{(4)}$$

where $H^{(0)} = \mathbb{D}^4$ and each $p$-handle $H_i^{(p)} = \mathbb{D}^p \times \mathbb{D}^{4-p}$ ($1 \leq p \leq 4$) is endowed with an embedding (called *attaching map*) $f_i^{(p)} : \partial \mathbb{D}^p \times \mathbb{D}^{4-p} \to \partial (H^{(0)} \cup \cdots (H_1^{(p-1)} \cup \cdots \cup H_{r_{p-1}}^{(p-1)}))$. 

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![Diagram of a handle-decomposition of a PL 4-manifold](https://via.placeholder.com/150)
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3- and 4-handles are uniquely attached to the union of 0, 1, 2-handles.
If \((\Gamma, \gamma)\) is a crystallization of a closed \(M^4\) and \(\{\{r, s, t\}, \{i, j\}\}\) is a partition of the five vertices of the associated pseudocomplex \(K(\Gamma)\), then \(M^4\) admits a decomposition of type

\[
M^4 = N(r, s, t) \cup_{\phi} N(i, j)
\]

where:

- \(N(r, s, t)\) denotes a regular neighborhood of the subcomplex of \(K(\Gamma)\) generated by vertices labelled by \(\{r, s, t\}\) (union of 0,1,2-handles)
- \(N(i, j)\) denotes a regular neighborhood of the subcomplex of \(K(\Gamma)\) generated by vertices labelled by \(\{i, j\}\) (union of 3,4-handles)
- \(\phi\) is a boundary identification.
Handle-decomposition of PL 4-manifolds

The hypotheses assumed about regular genus in many of the previous statements imply the associated handle-decomposition to lack in 2-handles; this fact allows to recognize the manifold $M^4$ as a connected sum of copies of $S^3 \times S^1$ and/or handlebodies.
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On the other hand, when at least a 2-handle appears, it is not possible to identify the represented 4-manifold, because of the great “freedom” in attaching 2-handles in dimension 4: the attaching map for a 2-handle in dimension 4 depends on a framed knot $(K, c)$, with $c \in \mathbb{Z}$.
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However, if the union of 0,1,2-handles is known to have spherical boundary, then the attachment of a unique 2-handle is proved to give rise to a $\mathbb{C}P^2$ component, via an important result of Gordon-Luecke.
Handle-decomposition of PL 4-manifolds

Note that exactly the above “freedom” concerning 2-handles yields to prove that in dimension $n = 4$ the classification of PL-manifolds is not finite-to-one with respect to regular genus.
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In fact, if $S^2 \times D^2$ denotes the trivial $D^2$-bundle over $S^2$ and $\xi_c$, for every $c \in \mathbb{Z} - \{0, +1, -1\}$, denotes the non-trivial one with Euler class $c$ and boundary $L(c, 1)$, then:

$$G(S^2 \times D^2) = G(\xi_c) = 3, \quad \forall c \in \mathbb{Z} - \{0, +1, -1\}.$$
Handle-decomposition of PL 4-manifolds

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In fact, if $\mathbb{S}^2 \times \mathbb{D}^2$ denotes the trivial $\mathbb{D}^2$-bundle over $\mathbb{S}^2$ and $\xi_c$, for every $c \in \mathbb{Z} - \{0, +1, -1\}$, denotes the non-trivial one with Euler class $c$ and boundary $L(c, 1)$, then:

\[ G(\mathbb{S}^2 \times \mathbb{D}^2) = G(\xi_c) = 3, \quad \forall c \in \mathbb{Z} - \{0, +1, -1\}. \]

It is an open question whether the number of PL 4-manifolds with fixed (possibly empty) boundary and fixed regular genus is finite or not.
Simple and semi-simple crystallizations

Definition ([Basak - Spreer, 2014]): A crystallization \((\Gamma, \gamma)\) of a closed PL 4-manifold \(M^4\) is simple if \(g_{ijk} = 1\) for all \(i, j, k \in \Delta^4\). Equivalently: if any pair of vertices of \(K(\Gamma)\) belongs to a unique 1-simplex (i.e. the 1-skeleton of \(K(\Gamma)\) coincides with the 1-skeleton of a single 4-simplex).

As a consequence: \(M^4\) is simply-connected.

Definition ([Basak - Casali, 2015]): A crystallization \((\Gamma, \gamma)\) of a closed PL 4-manifold \(M^4\) is semi-simple if \(g_{ijk} = m + 1\) for all \(i, j, k \in \Delta^4\), where \(m = \text{rk}(M^4)\). Equivalently: any pair of vertices of \(K(\Gamma)\) belongs to exactly \(m + 1\) 1-simplices.
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Simple and semi-simple crystallizations

Definition ([Basak - Spreer, 2014]):

A crystallization \((\Gamma, \gamma)\) of a closed PL 4-manifold \(M^4\) is \textit{simple} if \(g_{ijk} = 1\) \(\forall i, j, k \in \Delta_4\).
Equivalently: if any pair of vertices of \(K(\Gamma)\) belongs to a unique 1-simplex (i.e. the 1-skeleton of \(K(\Gamma)\) coincides with the 1-skeleton of a single 4-simplex).

As a consequence: \(M^4\) is simply-connected.

Definition ([Basak - Casali, 2015]):

A crystallization \((\Gamma, \gamma)\) of a closed PL 4-manifold \(M^4\) is \textit{semi-simple} if \(g_{ijk} = m + 1\) \(\forall i, j, k \in \Delta_4\), where \(m = rk(M^4)\).
Equivalently: any pair of vertices of \(K(\Gamma)\) belongs to exactly \(m + 1\) 1-simplices.
Characterization of manifolds admitting simple and semi-simple crystallizations

Simple and semi-simple crystallizations are proved to be “minimal” both with respect to gem-complexity and regular genus.
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[Casali - Cristofori - Gagliardi, 2015]

A closed simply-connected PL 4-manifold $M^4$ admits simple crystallizations if and only if $k(M^4) = 3\beta_2(M^4)$.

If $M$ admits simple crystallizations, then $G(M^4) = 2\beta_2(M^4)$. 
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[Basak - Casali, 2015]

A closed PL 4-manifold $M^4$ with $rk(M^4) = m$ admits semi-simple crystallizations if and only if $k(M^4) = 3\chi(M^4) + 10m - 6$.

If $M^4$ admits semi-simple crystallizations, then $G(M^4) = 2\chi(M^4) + 5m - 4$. 
MANIFOLDS ADMITTING \textit{SIMPLE CRYSTALLIZATIONS}:

\[ \mathbb{CP}^2, \quad S^2 \times S^2, \quad K3 \quad \text{and their connected sums} \]
Manifolds admitting simple and semi-simple crystallizations

MANIFOLDS ADMITTING *SIMPLE CRYSTALLIZATIONS*:

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MANIFOLDS ADMITTING *SEMI-SIMPLE CRYSTALLIZATIONS*:

\[ S^3 \times S^1, \quad S^3 \tilde{\times} S^1, \quad \mathbb{R}P^4 \quad \text{and their connected sums} \]
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\[ S^3 \times S^1, \quad S^3 \tilde{\times} S^1, \quad \mathbb{R}P^4 \quad \text{and their connected sums} \]

[Basak - Spreer, 2014] [Basak - Casali, 2015]

Let \( M^4 \) and \( M^4' \) be two PL 4-manifolds admitting simple (resp. semi-simple) crystallizations. Then, \( M^4 \# M^4' \) admits simple (resp. semi-simple) crystallizations, too.
Examples of simple crystallizations
Examples of simple crystallizations

A simple crystallization of $S^4$
Examples of simple crystallizations

A simple crystallization of $\mathbb{C}P^2$
Examples of simple crystallizations

A simple crystallization of $\mathbb{S}^2 \times \mathbb{S}^2$
Examples of semi-simple crystallizations

Semi-simple crystallizations of $S^3 \times S^1$ and $S^3 \tilde{\times} S^1$
Examples of semi-simple crystallizations

A semi-simple crystallization of $\mathbb{RP}^4$
Let $M^4$ and $M^4'$ be PL 4-manifolds admitting simple or semi-simple crystallizations. Then:

$$G(M^4 \# M^4') = G(M^4) + G(M^4')$$

and

$$k(M^4 \# M^4') = k(M^4) + k(M^4').$$

Consequence:

Let $M^4 \cong \text{PL}(\# p \mathbb{C}P^2 \# (\# p' \mathbb{C}P^2) \# (\# q \mathbb{S}2 \times \mathbb{S}2) \# (\# r \mathbb{S}3 \times \mathbb{S}1) \# (\# r' \mathbb{S}3 \hat{\times} \mathbb{S}1) \# (\# s \mathbb{R}P^4) \# (\# t \mathbb{K}3)$, with $p, p', q, r, s, t \geq 0$. Then,

$$k(M) = 3(p + p' + 2q + 22t) + 4(r + r') + 7s$$

and

$$G(M) = 2(p + p' + 2q + 22t) + r + r' + 3s.$$
Let $M^4$ and $M^{4'}$ be PL 4-manifolds admitting simple or semi-simple crystallizations. Then:

$$G(M^4 \# M^{4'}) = G(M^4) + G(M^{4'}) \quad \text{and} \quad k(M^4 \# M^{4'}) = k(M^4) + k(M^{4'}).$$
Let $M^4$ and $M^4'$ be PL 4-manifolds admitting simple or semi-simple crystallizations. Then:

$$G(M^4 \# M^4') = G(M^4) + G(M^4') \quad \text{and} \quad k(M^4 \# M^4') = k(M^4) + k(M^4').$$

Consequence:

Let $M^4 \cong_{PL} (\#_p \mathbb{C}P^2)\#(\#_{p'}(-\mathbb{C}P^2))\#(\#_q(S^2 \times S^2))\#(\#_r(S^3 \times S^1))\#(\#_{r'}(S^3 \times S^1))\#(\#_s\mathbb{R}P^4)\#(\#_tK3)$, with $p, p', q, r, s, t \geq 0$. Then,

$$k(M) = 3(p + p' + 2q + 22t) + 4(r + r') + 7s$$

$$G(M) = 2(p + p' + 2q + 22t) + r + r' + 3s.$$ 

In particular: $k(K3) = 66$ and $G(K3) = 44$. 

Maria Rita Casali

CRYSTALLIZATIONS OF PL 4-MANIFOLDS
Let $(\Gamma, \gamma)$ be a simple crystallization of a PL 4-manifold $M^4$. Then, for any partition \{\{i, j, k\}, \{r, s\}\} of $\Delta^4$, the coloured triangulation $K(\Gamma)$ of $M^4$ induces a handle decomposition of $M^4$ consisting of one 0-handle, $\beta_2(M^4) = g_{rs} - 1$ 2-handles and one 4-handle. $K(r, s)$ consists of exactly one 1-simplex (hence: $N(r, s) = D^4$). $K(i, j, k)$ consists of $g_{rs}^2$ simplices, all having the same boundary (hence: $N(i, j, k) = D^4 \cup (H^{2}(1) \cup \cdots \cup H^{2}(g_{rs} - 1))$).

Kirby Problem n. 50: Does any closed simply-connected 4-manifold admit a handlebody decomposition without 1- and 3-handles?
Let $(\Gamma, \gamma)$ be a simple crystallization of a PL 4-manifold $M^4$. Then, for any partition $\{\{i, j, k\}, \{r, s\}\}$ of $\Delta_4$, the coloured triangulation $K(\Gamma)$ of $M^4$ induces a handle decomposition of $M^4$ consisting of one 0-handle, $\beta_2(M^4) = g_{rs} - 1$ 2-handles and one 4-handle.
[Casali - Cristofori - Gagliardi, 2015]

Let \((\Gamma, \gamma)\) be a simple crystallization of a PL 4-manifold \(M^4\). Then, for any partition \(\{\{i,j,k\}, \{r,s\}\}\) of \(\Delta_4\), the coloured triangulation \(K(\Gamma)\) of \(M^4\) induces a handle decomposition of \(M^4\) consisting of one 0-handle, \(\beta_2(M^4) = g_{rs} - 1\) 2-handles and one 4-handle.

- \(K(r,s)\) consists of exactly one 1-simplex (hence: \(N(r, s) = \mathbb{D}^4\))
- \(K(i,j,k)\) consists of \(g_{rs}\) 2-simplices, all having the same boundary (hence: \(N(i,j,k) = \mathbb{D}^4 \cup (H_{1}^{(2)} \cup \cdots \cup H_{g_{rs} - 1}^{(2)})\))
Let \((\Gamma, \gamma)\) be a simple crystallization of a PL 4-manifold \(M^4\). Then, for any partition \(\{\{i, j, k\}, \{r, s\}\}\) of \(\Delta_4\), the coloured triangulation \(K(\Gamma)\) of \(M^4\) induces a handle decomposition of \(M^4\) consisting of one 0-handle, one 2-handles and one 4-handle.

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**Kirby Problem n. 50:** Does any closed simply-connected 4-manifold admit a handlebody decomposition without 1- and 3-handles?
Exotic structures and simple crystallizations

[Casali - Cristofori, 2015]

Let $M^4$ and $M'^4$ be two closed PL 4-manifolds, with $M^4 \cong_{TOP} M'^4$. If both $M^4$ and $M'^4$ admit simple crystallizations, then $k(M^4) = k(M'^4)$. 
Exotic structures and simple crystallizations

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Let $M^4$ and $M^4'$ be two closed PL 4-manifolds, with $M^4 \cong_{TOP} M^4'$. If both $M^4$ and $M^4'$ admit simple crystallizations, then $k(M^4) = k(M^4')$.

Consequences:

- Let $M^4$ be $S^4$ or $\mathbb{CP}^2$ or $S^2 \times S^2$ or $\mathbb{CP}^2 \# \mathbb{CP}^2$ or $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$; if an exotic PL-structure on $M^4$ exists, then the corresponding PL-manifold does not admit simple crystallizations.

- If $r \in \{3, 5, 7, 9, 11, 13\} \cup \{r = 4n - 1/n \geq 4\} \cup \{r = 4n - 2/n \geq 23\}$, then infinitely many simply-connected PL 4-manifolds with $\beta_2 = r$ do not admit simple crystallizations.

- Let $\bar{M}$ be a PL 4-manifold TOP-homeomorphic but not PL-homeomorphic to $\mathbb{CP}^2 \#_2 (-\mathbb{CP}^2)$; then, either $\bar{M}$ does not admit simple crystallizations or $\bar{M}$ admits an order 20 simple crystallization.
An answer to this open question could arise from the 4-dimensional catalogue of order 20 crystallizations, whose analysis is currently underway!
Exotic structures and simple crystallizations

An answer to this open question could arise from the 4-dimensional catalogue of order 20 crystallizations, whose analysis is currently underway!

Note that, in case of positive answer, we would have the first example of a simple crystallization of an exotic PL 4-manifold.