Graphic Calculators and Micro-Straightness: Analysis of a Didactic Engineering

Michela Maschietto

Abstract This paper concerns the analysis of a didactic engineering, the aim of which is to introduce Calculus, at secondary-school level, through the relationship between global and local points of view. It was designed for a graphic–symbolic calculator environment and structured in accordance with a learning trajectory from identifying the graphical phenomenon of local linearity to its mathematical formulation. This learning trajectory involves the reconstruction of the relationship with the tangent line to a curve at a chosen point. The analysis shows the use of different semiotic systems in order to grasp this phenomenon and construct its mathematical meaning.

Keywords Calculus, Graphic–symbolic calculator, Didactic engineering, Linearity, Gesture

1 Introduction

Question: What do you think of this experience that you have been exposed to in class?

A1. "It's interesting because it stimulates our curiosity to go and poke our noses into the points of a curve without leaving them any privacy."

Question: What role did the calculator have in the activities carried out?

A2. "The calculator was our magnifying lens. It was essential for "unearthing" the micro-straightness. Without it the whole work would have been impossible."

The above questions and answers are taken from a questionnaire given out at the end of a teaching experiment carried out within a research project on the introduction of Calculus in secondary schools (Maschietto 2002). The answers of the two students, A1 and A2, seem to clearly highlight some characteristic elements of this project:

– the artefacts (graphic–symbolic calculators) used in the classroom activities and, in particular, the controls performed (the zoom);
– the fundamental role of the calculator ("it was essential");
– the type of task proposed, requiring the exploration of graphical representations of functions around chosen points ("of a curve without leaving them any privacy") and leading from a global point of view to a local one (an aspect of the "global/local game");

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– the property emerging from the exploration of the functions (the phenomenon of local linearity “micro-straightness”), and the mathematization process carried out;
– the attention paid to the process of the construction of meaning through the explorations carried out and the zooming process (“calculator was our magnifying lens”); and, finally,
– the fact that a positive evaluation emerges from the work carried out.

In this article, the above-mentioned elements will be discussed in order to show how the construction of a mathematical meaning of “micro-straightness” can be supported by a specific approach. The research hypothesis is that the transformations of the graphical representation of a function through the use of zoom-controls and the experience of the perceptive phenomena of “micro-straightness” that these transformations provoke, can give rise to the formulation of some specific language, the construction of metaphors and the production of gestures and specific signs by the students. Our hypothesis is also that adequately exploited by the teacher, these germs can lead to an entrance in the local/global game, fundamental in Calculus and analysis hardly observed at high-school level.

In the first part of the article, we present the main theoretical elements underlying this project which have both influenced the design of the proposed set of tasks and the analysis of their implementation. In fact the research project relies on constructs coming from the theory of didactic situations (as it is structured according to the definition of didactic engineering), the theory of semiotic mediation and study on gestures, combing these in a coherent way. In the second part, we then present some elements of the analysis of the three experiments carried out and the results obtained.

2 Theoretical Background

Studies on the didactics of Calculus, which have been carried out for many years (Artigue 1998), have stressed the many obstacles and difficulties encountered by students in approaching this field, not only at secondary-school level but also at university. The research project presented in this article is based not only on those results, but also on some characteristics of Calculus compared with other mathematical fields (Artigue 1996). In particular, we focus on one of these characteristics—the necessary interaction between different points of view what we call the “global/local game” and analyse in the section “Epistemological Component: The Global/Local Game”.

The idea of using graphic manipulations, such as enlargements, in order to introduce mathematical notions (like the notions of derivative and/or tangent line), which is not new in mathematics education, can be interpreted as a way of engaging students in this global/local game. Several research projects have exploited it in pencil-and-paper work (for instance, Sierpinska 1985 or Groupe 1999) and ICT environments, for instance, with calculators (Artigue 2002) or specific software (Tall 2000, 2003). In this article, we shall delve into these studies.

In this study, three components have been considered (Arzarello and Bartolini Bussi 1998):

– an epistemological component within which the global/local game is analysed;
– a cognitive component, focussing on studies on the role of gestures and metaphors in mathematical conceptualizations and learning processes; and
– a didactic component, concerning the construction of didactic engineering for initiating the local/global game.

2.1 Epistemological Component: The Global/Local Game

Even before Calculus is taught at secondary-school level, students meet functions as algebraic and geometrical objects mainly from two points of view—a global point of view and a pointwise point of view.

A global point of view on functions (and, similarly, a pointwise one) presents two aspects. The first aspect involves the consideration of functions as entities defined by one or several formulas and/or graphical representations (which can be considered prototypic). The second aspect concerns their properties. A property of a function f is global if it is a property satisfied by f in a given subset of its domain. A pointwise point of view for its part, considers a function from the values taken at one or several chosen points, belonging to its domain (for instance, like in a table). Here is an example: the function f defined as follows $f : \Re \to \Re$, $f(x) = x^2 - 1$. From a global point of view, f is known as a polynomial function of degree two whose graphical representation is a parabola. For this function, the property "the function f is positive at $x = 2$" is a pointwise property, whereas the property $f \geq 0$ on $[1,+\infty[$ is a global property.

Since the beginning, the study of Calculus is characterised by the addition of a local point of view on functions. Some authors also distinguish an infinitesimal point of view, present when properties involve infinitesimal notions (Chorlay 2007). A local (and infinitesimal) point of view draws attention to what happens at a specific value and in its neighbourhoods. The pointwise and local points of view (and also the infinitesimal one) are difficult to separate from each other. This makes their interaction complex. With respect to the function f mentioned above, the statement: "the function f is differentiable at $x = 2$" corresponds to a local property. One cannot make sense of it by only knowing $f(x)$ for $x = 2$. It is a local (and infinitesimal) property but with a pointwise statement.

In the graphic register, a local point of view draws attention to what happens around a chosen point. In the given example above, the curve representing the function f, enlarged around the chosen point, is well approximated (at order one) by a straight line or, in other words, becomes "locally straight" or "micro-straight".

This example shows what the students must confront with respect to the linearity when the notion of differentiable function at a point is introduced. Linearity was previously seen as a global phenomenon, associated with a particular class of functions defined on intervals (or on $\Re$). It has then to assume local features. This change also demands the reconstruction of the relationship with the notion of tangent line to a curve at a point. Several researchers have pointed out the difficulty of this process (see for instance Castela 1995).

2.2 Cognitive Component: Metaphors, Gestures, and Signs

Introducing the idea of local linearity to students from the beginning of Calculus was suggested by Tall and his colleagues many years ago (e.g. Tall 1989). In particular, in their work they distinguish between "local straightness" (the property of some graphical representations to seem straighter and straighter when zooms around their points are successively performed) and "local linearity" (the property of some functions to be approximated by a linear function in the neighbourhood of a point). Local straightness is considered by Tall as a cognitive root for local linearity. Tall further clarifies this by remarking that local straightness involves a perceptive level and evokes a global approach, while local linearity focuses on the best linear approximation of a function at a point, expressed formally. These elements were implemented using specific software (Graphic

Calculus, 1986) and afterwards revisited within the embodiment approach (Tall 2000; Gray and Tall 2001).

Our research work on the global/local game also takes into account research work concerning embodied cognition, and more specially the role played by conceptual metaphors in the construction of mathematical concepts (for instance, Lakoff and Nu´n˜ez 2000). These have stressed, in particular, the relevance of the analysis of linguistic expressions and accompanying gestures. The role of gestures is analysed in several studies (see for instance, the classification proposed in McNeill 1992; Arzarello and Edwards 2005). From a different point of view, other research stresses the relationships between the perceptive and abstract levels. For example, in his cognitive and epistemological approach to the question of foundation in mathematics, Longo (1997, 1998) attributes great importance to both the subject's interactions with the world (perceptions, spatial relationships, and movement) during the process of conceptualisation and reciprocal influences between the perceptive (phenomenological level) and mathematical levels (where concepts are formalised) in doing mathematics.

These works, with other research work concerning neuroscience, increasingly influences didactic research on the analysis of cognitive processes involved in mathematical conceptualisations (Arzarello and Edwards 2005). In their research, Radford et al. (2005) for their part emphasize the role of body and signs in thinking and learning processes. They stress that "thinking is not only mediated by, but also located in, body, artefacts, and signs". Within a Vygotskijan perspective (Bartolini Bussi et al. 2005; Bartolini Bussi and Mariotti 2008), gestures emerging from interaction with an artefact are considered as a semiotic system which has to be considered together with other systems, like language, graphic signs, etc. Within this perspective, the shift from a perceptive level to an abstract one can be regarded as a move from the level of manipulation of the artefact to the level of mathematics, where the mathematical content related to the artefact is expressed as a part of mathematical knowledge. Within this framework, Falcade et al. (2007) propose the analysis of Cabri (Dynamic Geometry Software) tools, for example Dragging and Trace, to construct mathematical meanings of the notion of function (in particular, variation and covariation).

All these research projects highlight that it is important to solicit, as a starting point, perceptual components in order to allow the construction of metaphors, but also gestures and signs by students. These elements, developed by students during selected activities, are available to the teacher, who can use them during the collective process of construction of mathematical meanings. These elements deeply influenced the didactic choices made for the design in our research.

Let us add that, since the calculator is a complex artefact, the interaction with this artefact will be considered in terms of interaction with specific commands, through which the meaning of the global/local game is mediated.

2.3 Didactic Component: the Didactic Engineering

The research hypothesis formulated in the introduction of this article stems from the theoretical considerations that transformations of the graphical representation of a function through the use of zoom controls and experience of the perceptive phenomena of micro-straightness can give rise to the formulation of specific language, the construction of metaphors, and the production of gestures and specific signs by students. These elements can be productively exploited by the teacher for initiating the local/global game at high - school level.

In fact, the above-mentioned research suggests that such an experience allows not only a certain phenomenon to be encountered, but also its invariant components to be selected. In this case, micro-straightness can play the role of invariant in the zoom processes and support the initiation of the global/local game.

For testing this hypothesis, diagnostic “didactic engineering”[1] (Artigue 1988; Brousseau 1997) is developed with the objective of introducing the global/local game through micro-straightness. The diagnostic status is linked to the fact that didactic engineering is chosen not only as a design methodology for the sessions but also as a methodology for studying how the global/local game can be achieved in class sessions and for exploring students’ cognitive processes.

In the a priori analysis phase of the didactic engineering, all the sessions were analysed, discussed and improved with the help of the teachers of the classes involved. In this analysis, as is usual, the didactic variables of the situations at stake were identified; values for these were selected taking into account their potential impact on students’ strategies and constructions. In coherence with the theory of didactic situations underlying didactic engineering, tasks and didactic variables were designed in order:

– to maximize the *a-didactic* potential of the situations (Brousseau 1997), in other words the potential of mathematical autonomy of the students; and
– to ensure that the expected outcome, the initiation of the local/global game, would be fostered by the interaction with the *a-didactic milieu*.

Nevertheless, taking into account the results of previous research (Artigue 2004) it was anticipated that the teacher would have a substantial role to play in the mathematization underlying the local/global game, and the affordances of the theory of semiotic mediations were used in order to plan this role. Each session was thus divided into two parts—first group work then collective discussion orchestrated by the teacher. Such a construction allowed us to play in a productive and coherent way on the respective and complementary affordances of the theory of didactic situations and theory of semiotic mediations. Evident similarities can be observed in the doctoral dissertation by Falcade (2006), and resonance with recent research on the networking and integration of theoretical frames (Kidron et al. 2008; Artigue 2007).

3 The Methodology

As we have considered above, didactic engineering represents the methodology of our research.

The observation of the sessions is organised as follows. One group is observed by means of a grid based on the a-priori analysis whereas another is filmed. The collective discussions are recorded. The three experiments have part of the sessions in common and some other sessions. The analysis contained in this article regards those parts that are common to all three experiments carried out, even if each of these is continued for a greater number of sessions than the ones considered. An evaluation is planned for each experiment (questionnaire, pre-test and/or post-test).

[1] Didactic engineering comprises three phases:
1. Preliminary analysis, in which the theoretical framework and aims of the research are established.
2. Construction and a-priori analysis.
3. Experiments, a-posteriori analysis, and conclusions. The validation of hypothesis is based on comparison of a-priori analysis and a-posteriori analysis.

4 The Experiments and Analyses

The experiments were carried out in fourth-year scientific high school classes (18-year-old students). Hereafter we refer to the experiments by the terms Exp_A, Exp_B, and Exp_C. In all three cases, each student was provided with a calculator (TI-89 or TI-92). The students are divided into pairs or groups of three and could only use one calculator per group (except the Exp_C students, who had been working with graphic–symbolic calculators for some time at school—the students in the other two classes were lent the calculators for the duration of the experiment).

This section is divided into two parts. In the first part, the characteristic elements of the sessions emerging from their a-priori analysis are presented. In the second part the three experiments are analysed. This a-posteriori analysis allows the formulated hypothesis to be tested and the choices made for the management of the activities to be checked. However, as is usual, this will show not only elements that comply with the a priori analysis but also some unexpected events in both the first and the second sessions.

4.1 Elements of the A-Priori Analysis

The activities proposed for the group-work and collective discussions were organised (in terms of time and work distribution) in accordance with the above-mentioned cognitive studies. In particular, we shared the position (Boero et al. 2001) according to which a metaphor could emerge from the students' private spaces and spread in the class as an object linked with gestures and specific language.

The objective of the first session is to lead the students to discover the microstraightness phenomenon. According to the a-priori analysis, the first session introduces the global/local game based on various components and intense semiotic activity. This session plan is designed to make the following available to the students to help them with the construction of the meaning of micro-straightness:

- a procedure mediated by the calculator to move from global to local points of view and vice versa;
- linguistic expression to indicate the zooming process or the result of this process;
- a graphical representation, potentially iconic, for the result of this process;
- gestural and metaphorical representations of the process and/or its result.

Encountering the micro-straightness phenomenon is achieved by means of the exploration of graphs representing selected functions around given points, using the zoom-controls available in the GRAPH application of the calculator. The proposed worksheet for the groupwork contains the following elements:

- determination of the values taken by six functions;
- graphical representations of the functions, first in the calculator screen and then on the worksheet;
- explorations of these graphical representations by using zoom-controls;
- graphical representations of the enlarged curves on the worksheet; and
- comparison among the different explorations.

The beginning of the exploration is guided in order to take into account the students' instrumentation level and to allow the construction of schemes for the use of the artefact (Artigue 2002). The choice of functions and points (some functions are differentiable over the whole domain, some others have singular points) is made to allow the students to:

– encounter micro-straightness during the first exploration;
– give the micro-straightness an invariant status for the exploration (for the functions differentiated at the chosen points) and use this invariant as an end-point in the subsequent explorations;
– encounter a counter-example, so that the local nature of the straightness is emphasised even more (this is done in order to question possible over-generalisation, such as "straightness is true for all the functions and/or all the points") and test the invariant in the case of a linear function, in terms of anticipating the results of the magnification.

In the worksheet for the group-work, some control methods for moving from the global point of view to a local one are suggested (that is, the zooming process control), using various requests:

– draw what is displayed in the standard window (ZOOMSTD, [-10, 10] $\times$ [-10, 10]) on a sheet of paper (first stage), then draw what is achieved after two zooms[2] (second stage) and, finally, what appears at the end of the exploration (third stage);
– always represent a constant function, to which most of the given points belong (in that way a new graphical reference is provided); and
– note the dimensions of the window represented (that is the intervals displayed) using the WINDOWS application (numeric control of the magnification).

The first point referred to above allows the work carried out to be tracked. In fact, while the linearity depends essentially on the magnifying process carried out by the calculator, the local nature of the phenomenon is made to emerge by two comparisons within a pencil-and-paper environment. The first comparison regards the transformation of the graph of the function examined by the standard window around a point of the graph (before and after the zooms). The second is a transverse comparison of the graphical representations of different functions after the subsequent magnifications around different points, thanks to the presence of functions that are not differentiable at the chosen points. The first point implies the joining of the two working environments, calculator and pencil and paper, that have different ways of treating the graphical representations.

The discussion during the collective phase, that follows the group-work, has various objectives–to let the students' interpretations emerge and to highlight the two components of the phenomenon mentioned, by comparing the work done by the various groups and by composing a common language. In particular, the search for a term to refer to the experience is elicited; this term then becomes an intermediary sign that the teacher can use to move from the work with the artefact (perceptive level) towards the mathematization process of the phenomenon (mathematical level).

As already mentioned, the group-work stage has a strong *a-didactic* component (Brousseau 1997), which is consistent with the notion of didactic engineering itself. In general, the language constructed around these explorations is important for the continuation of the experiment, as it provides clues for the conceptualisation process being carried out by the students. During this first stage, the teacher only intervenes for technological support if needed (that way, the instrumentation level of the students is taken into account) or to clarify the task. His/her role becomes more important during the collective discussion stage, as the a posteriori analyses confirm.

[2] ZOOMIN or ZOOMBOX.

The elements constructed during the first session lead to the second session, whose objectives are to mathematize the micro-straightness phenomenon, reconstruct the relationship with the tangent of a curve at a point, and introduce a new calculation based on orders of magnitude.

During the group-work, the students are asked to check graphically that an assigned function verifies the condition of micro-straightness (around the point of abscissa $x = 2.5$), identified during the first session (in the worksheet we have written the term previously introduced in class). The students are then asked to calculate the equation of the line that appears on the calculator display, which in other words means determining an approximation of the function around the given point. Two strategies are planned for–a graphic strategy, where the Cartesian coordinates of two points on the line displayed are searched using the cursor or the TRACE control, and a numeric strategy, where the coordinates of the points are chosen from a table of values of the function (for example, moving to application TABLE). The next collective phase is based on the hypothesis that the students will obtain different equations, because the zooming process term can correspond with different-sized windows. Comparison of the various equations and return to the standard window (or one of a similar size) in which the lines that are represented (or almost, this depends on the calculations) all seem to be near to the curve and cannot be distinguished from each other must allow the graphic phenomenon to be made into a problem. So, the need to go beyond the perceptive level is elicited. The discussion must therefore leave the graphic register and initiate work in the symbolic register. To advance the mathematization process, the abscissa value of the second point on the curve can be written as $x = 2.5 + h$ or $x = 2.5 - h$ (where $h > 0$) and the iteration of the zooms can be associated with the idea of taking smaller and smaller h values. The shift from the approximations to the function that represents the tangent is mathematically linked to the calculation of the limit of the gradient of the approximations for h towards zero. However, the calculation in this case is based on the language developed during the sessions, and drawing on the orders of magnitude (relative and/or absolute; Deledicq and Diener 1989) rather than on the explicit introduction of the notion of limit. The mathematical object thus constructed is an ideal object, the tangent, that encapsulates the approximation process (this is an infinite process, suggested on the worksheet for session 1 by the expression "subsequent zooms" and it refers to the metaphor for infinity; Lakoff and Núñez 2000). On the basis of what has been described above, it was predicted that the management of this collective phase would be delicate–it needs to make the students feel aware of the necessity of the mathematization process, to adequately support the transition from the numeric to the symbolic and the interpretation of the zoom process in symbolic terms, without losing their participation and interest.

The third session deals with re-using the calculation that has just been encountered with the orders of magnitude, by means of a request to determine the equation of lines that are tangent to given curves. Therefore, the third session is presented as a session for technical work (Artigue 2002) where the application of symbolic calculation is drawn upon.

4.2 Elements of the A-Posteriori Analysis: First Session

In this part the three experiments are analysed, highlighting not only the elements that did comply with the a-priori analysis, but also the unexpected ones.

4.2.1 Linear Invariant and Local Nature

The task allows the global/local game to be introduced, at least as far as the treatment of the graphical representations is concerned (see the section “Elements of the A-Priori Analysis”).

Excerpt 1 DAL-DF-MA group (Exp_A)

15. DF: “Forward zoom” (*he carries out the 3rd ZOOMIN*)
16. DF: “Again” (he carries out the 4th ZoomIn)
17. DF: “It becomes straighter and straighter”
18. DF: “The drawing is the same as before. Even if the result is the same, we’ll write it down”.

After getting the representation in the standard window, DF does 2 ZOOMIN S

20. DF: “I want the other piece of function. It’s still a line! Draw at least one axis” (*addressed to MA. DF carries out the 3rd ZOOMIN*)
21. DF: “We’ll stop here because it stays the same”.

In the pencil-and-paper environment (Fig. 1b), the linearity is emphasised by the use of a ruler to draw the graphical representation that appears on the calculator display on the third sheet (end of the exploration).

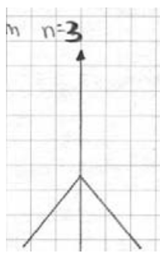
a b

Fig. 1 Window at the end of the exploration process (Exp_A)

In other protocols (Exp_B and Exp_C), the students try to explain the end-point of their exploration, for example: “REASON WHY WE STOPPED CARRYING OUT THE ZOOMS ? *The more we used the ZoomIn, the more the curve sector considered tended to become a line*”. We observe here a dynamic language, that draws on the infinite approximation process.

In the protocols, there are two distinct phenomena, linked to the local point of view. The first regards the strength of the “straight” nature at a perceptive level. The second regards the interference of the global point of view with the local one. As far as the first phenomenon is concerned, the comments (for example, Excerpt 2) on the exploration of the corner (function y3[3]) highlight that at this stage the students have, in general, clearly identified the graphic phenomenon “it becomes straight using the zoom”.

Excerpt 2. DAL-DF-MA group (Exp_A)



In all these cases the functions, even with the second zoom, are similar to a line with a gradient ≥ 0 but:
- y4 is similar to a line only after the 4th zoom [Note: at $x = 1/\pi$]
- y3 is similar to two lines (one with $m > 0$ and the other with $m < 0$)

[3] y3(x) = $-x^3 - 2|x| + 4$ at $x = 0$

However, this recognition does not allow them to distinguish the situation of the function that is differentiable at the given point and that of the function having two different half derivatives and leading to a corner. In fact, these situations, mathematically different, are unified by their common "straightness" recognized at a perceptive level (Excerpt 2). The second function does not therefore represent a counter-example, unlike what is hypothesised in the a-priori analysis. Their distinction will only occur during the mathematization process of the linear invariant. The real counter-example is provided by the y4[4] function, the graphical representation of which, after subsequent zooms, is perceptively different. In this case there is no move from the "curve" category to the "straight" category, as happens for all the other functions.

As far as the second phenomenon is concerned, interference of the global point of view can be identified in the protocols, and there is some resistance in the move to a local point of view, above all in the pencil-and-paper environment. This is, for instance, visible in the excerpt reproduced below (Excerpt 3).

Excerpt 3. CF-GL-CA group (Exp_A)

The students analyse the second function y2(x) = $\frac{8x+1}{(2x-1)^2}$

31. CF: "Enlarge it" (*he addresses this to Angelo, who does two ZoomIns*).
32. GL: "Come on! It's a line!". (…)
39. CF: "there are no graphical references".

From the excerpt it can be seen that the result of the magnifications starts to become clear to student GL (No. 32), who checks the process on the calculator: the disappearance of the graphic references (axes of Cartesian coordinates) in the subsequent zooms does not seem to interfere with this process. However, when the production of drawings on the sheet is required, that is, to move from the "new" environment to a well-known one, the question of the references becomes of primary importance for the student CF, who takes charge of it. In CF's utterance (No. 39), we observe the importance given to the Cartesian axes as element of reference, linked to a global point of view on the function: the student does not use the references suggested by the worksheet (the constant function passing through the point around which the zooms are carried out). In fact, this line is only drawn for the first function explored (Fig. 1) and the dimensions of the windows after the various zooms are not indicated. The persistence of this global point of view can also be found in the third representation in which a vertical line appears on the right, which is the trace of the axes present in the first and second drawings (Fig. 2).

As far as the involvement of the students is concerned, Excerpt 3 shows a clear difference between those who enthusiastically manage the calculator and those who, on the other hand, seem to be burdened with the task of representing the graphs on the sheets of paper.

[4] $y4(x) = \begin{cases} 4 + x\sin\frac{1}{x} & \text{at } x \neq 0 \\ 4 & \text{at } x = 0 \end{cases}$

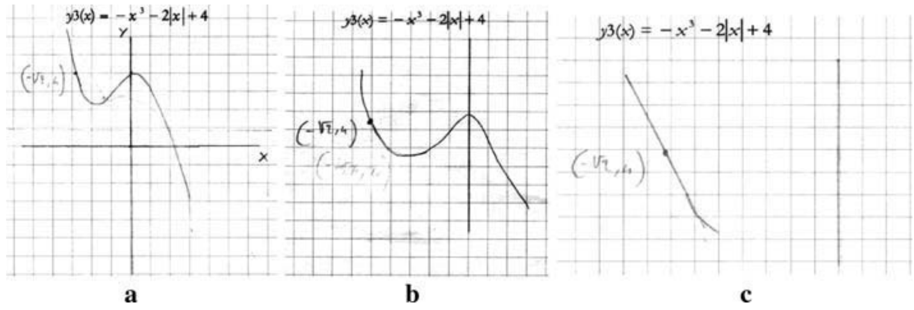


Fig. 2 CF-GL-CA group (Exp_A)

In general, the students involved in the various tasks do not trace the constant function required by the worksheet text or, if they trace it, they do not seem to use it as a new graphic reference element. Besides, in most of the protocols the sizes of the windows are not present.

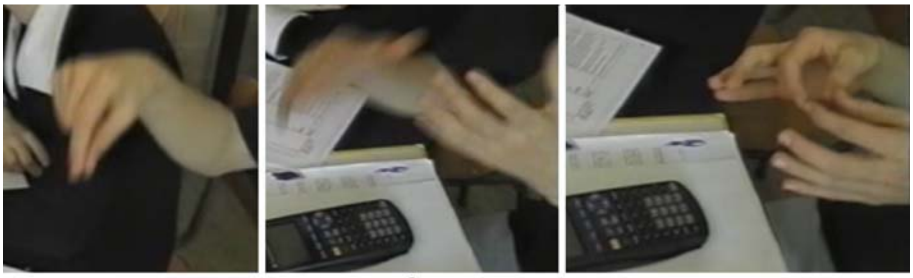
This suggests that the local nature of the invariant is a property that could disappear if it is not supported by appropriate signs (or ostensives according to Bosch and Chevallard 1999) and the action of the teacher.

4.2.2 Gestures and Metaphors

In accordance with the a-priori analysis, the activity presented to the students shows its potential for the production of gestures and metaphors. These appeared both during the communication inside the groups and during the collective discussions. The analysis of the students' protocols and the discussions show that the conceptualisation of the zoom-controls, that supports the localisation of the view, appears through gestures that accompany the explanation of the exploration strategies and linguistic expressions that can be analysed in terms of metaphors.

A particularly representative example is the analysis of the gestures of one student, PM (Exp_A), while he is explaining the exploration of a graphical representation. The ZoomIn control is used in order to see some of the characteristics of the curve in a detailed way and is associated with a downward movement meaning an "entrance into the curve", that corresponds with moving into the curve (*ZoomIn gesture*, Fig. 3a). The ZoomOut control, which is used to obtain a bigger curve and to study its characteristics better, is associated with an upward movement meaning an "exit from the curve" (*ZoomOut gesture*, Fig. 3b), which also corresponds with moving away from the curve. PM's gestures lead the details of the curve to be interpreted as downwards and the overall curve as upwards. PM also creates a space in front of him for controlling these processes (the standard window of the calculator becomes a little rectangle that is constructed by his fingers, Fig. 3c).

The reference to the ZoomOut control identifies the space under his eyes, while the palm of one hand is associated with the flat part that is obtained from the ZoomIn. In this way, PM has created his own space, which is suggested by the activity with the calculator, where the two different transformations of the curve can co-exist and be controlled.



a b c

Fig. 3 PA's gestures (Exp_A): ZoomIn, ZoomOut, standard window

During the collective discussions, the teacher and students share and exchange interpretations of the zooming process. As expected, gestures and metaphors appear in them. In particular, the metaphor "the calculator is a microscope" is identified, to which two other elements are linked: the zooms let you enter the graph and they let you obtain a line. For example, in one class, expressions like "reduce, go inside the graph" were encountered that are similar to expressions like "go and see a piece of curve, go very close" that were present in other classes. The language "very close" and "very small" seem to be similar to the analysis done by Lakoff and Núñez (2000) on the conceptualisation of the continuum and is also similar to what was identified as proximity metaphors. The expressions used are associated with gestures, which join the *ZoomIn gesture* to that of proximity (the fingers of the two hands are close to each other and they are pointing downwards: *ZoomIn-proximity gesture*) or which are similar to the previously described zooming process (the fingers of the two hands come together and move forwards once and backwards once: *zoom gesture*)

Besides what has just been analysed, another metaphor seems to appear, which is that of "change of scale". To deal with the move from the "curved" category to the "straight" category, the teacher (Exp_B) first elicits a change in space (from the two dimensions of the graphical representation to the three dimensions of the surrounding environment): the point of the function becomes the student and the curve on which the point is becomes a sphere on which the student stands. Then, she suggests changes in scale ratio through which the dimension of the student is reduced relative to that of the sphere. From the student's point of view the perception of the spherical surface changes (TH[5]: "Zoom means to enlarge (*starting from the desk, she moves his hands upwards*) a portion or to reduce us (*her hands move downwards*) inside the function and go in, to see what happens (…)"). This metaphor seems interesting, as it turns the change from curve to line into an experience that the students can conceptualise in another environment and positions the local point of view in the subject himself. It is successfully used in the case of the corner, for which another gesture appears (*corner gesture*, Excerpt 4).

Excerpt 4. Exp_A

C33. TH: "It's a bit like that situation where I am at the top of Cervino[6], but exactly… right at the very top. (…) If I am at the top of a very pointed mountain, but really pointed (*her hands make a point*), I can make myself as small (*the distance between her fingers reduces*) as I want, this corner…"

C34. A: "It is still a corner".

However, this introduces some difficult elements with respect to the session's objective, as

[5] Teacher.

[6] Italian mountain.

it leads the students to three-dimensions and to consider planes rather than lines ("Illusory planes…").

All the elements that emerge from the collective discussions are available for the teacher to advance the construction of meaning for the micro-straightness and start the process that is planned for the second session.

4.2.3 Language

During the explorations, the students' language is enriched with terms that are strictly linked to what they are experiencing. These appear not only as an answer to the teacher's questioning, in accordance with the a-priori analysis, but also spontaneously in the interpretation of the graphical representations displayed on the calculator screen.

As far as the request to name the phenomenon that has been discovered is concerned, during the collective discussion of Exp_A, after several exchanges and invitations by the teacher, the students agreed on the expression "zoomata lineare"[7], which underlines the idea of the enlargement process and its result ("linear"). In the above-mentioned expression, the importance attributed to the local nature of the phenomenon is not obvious. At a later time, a graphical representation is associated with this term (Fig. 4). This sign has all the potential of an iconic representation of micro-straightness (*sign of micro-straightness*), where the horizontal line serves to find the point in question. As shown by previous research, this type of representation plays an important role in the field of Calculus (Maschietto 2001).

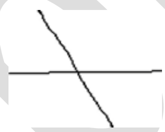


Fig. 4 Representation of "zoomata lineare" on the blackboard

In the collective discussion of Exp_C, this phase was very rich. At the beginning, the students proposed some terms, for example "segmentizzazione"[8] or "segmentizzata"[9], where the word "segment" can be recognised as the root. This choice could be linked to the interpretation of the graphical representation on the calculator screen. Finally, the shared expression contained the prefix "micro", introduced by the teacher to draw attention to the local nature.

The management of the session seems to have allowed the students to construct their own interpretations of both the phenomenon that they were exploring and some graphical representations they had obtained. In particular, besides micro-straightness, on different occasions the students expressed their interpretations during the group-work. These interpretations become more visible when the students are encouraged by the teacher to compare their ideas with those of their classmates. In this case, the role of the teacher is crucial, because she/he forces those ideas to come out of students' individual spaces.

For example, with respect to function y4, during the group-work, the student CF (Exp_A) says: "In the fourth case we get an electrocardiogram!" or GL (Exp_A): "The more zooms you do, the more jagged it becomes!". At the beginning of the second session,

[7] Italian word.

[8] Italian word.

[9] Italian word.

the same student CF recalls the function and its behaviour. Besides, when presenting the exploration of this function, he introduced a graphic sign (*zoom sign*, that was also present in the final evaluation test) to understand the procedure being followed and its result (Fig. 5).

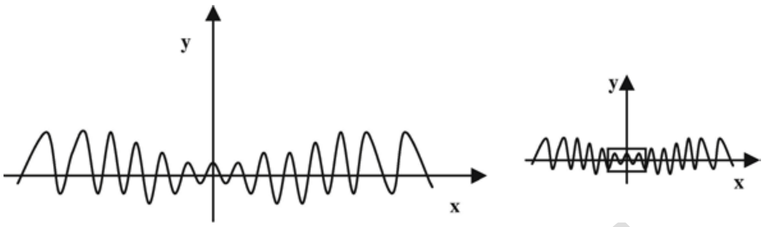


Fig. 5 CF’s graphical representations on the blackboard (Exp_A)

The analysis shows that the management of the process to name what has been obtained is very delicate and that the students are initially a bit reticent. However, the protocols show that the students name and interpret the representations that they obtain from the calculator. This fact can therefore be interpreted in terms of responsibility for the creation of language and/or introduction of new terms that have an official use. In the classes considered here, these are not the responsibility of the students, but those of the teacher only.

The analysis of the second session shows not only how the characteristic elements of the first session’s work are revised, but also the emergence of new phenomena, as already mentioned at the beginning of this part.

4.3 Elements of the a Posteriori Analysis: Second Session

As already mentioned in the section “Elements of the A-Priori Analysis”, the second session includes a first phase of group-work where the students are led to deal with microstraightness graphically and numerically, by means of the request to determine the equaion of the line that appears on the calculator screen, and a second collective phase where the results are compared and there is a move from numeric to symbolic.

In the group-work protocols two strategies for determination of the equation for the identified line, a graphic strategy and a numeric strategy, have been identified in the a-priori analysis. As expected, the equations provided by the students depend on the chosen points and are different for each group.

Alongside the planned strategies, another strategy appears, linked to the recognition of the line to be determined as the tangent to the curve at the given point. This encourages the students to set the algebraic procedure[10] from the previous study of conics sections. This strategy was not anticipated in the a-priori analysis, as it was considered that this link with the tangent could only emerge in the graphic register if the reciprocal positions of lines and curve were considered from a global point of view. On the worksheet prepared for the group-work, the students are in fact encouraged to work on the enlarged part of the graph and a return to the global point of view is not planned. However, the analysis of the

[10] Steps:

1. Set the system with the equation of the curve and that of the line passing through the given point (depending on a parameter).
2. Reduce the system to a second-degree equation with a parameter.
3. Compute the parameter imposing the condition for a square root.

resolutions of the groups having adopted the algebraic strategy leads to interpret the recognition of the tangent in terms of spontaneous connection between the local and global points of view, based on the comparison of two different graphical representations as seen in the protocol described above (Fig. 6).

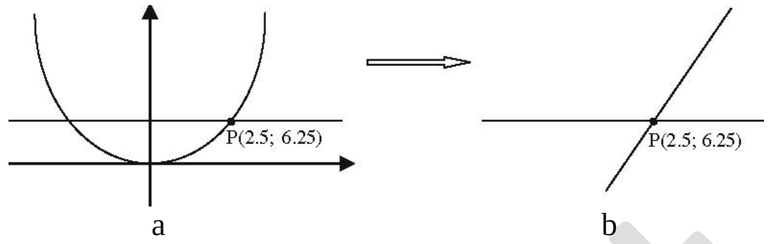


Fig. 6 Graphical representations on the worksheet, CF-GL-CA group (Exp_A)

In this protocol, alongside the standard representation of the function (a parabola in Exp_A) required by the worksheet, the student CF places the result of the zooming process. A comparison or overlapping of the two drawings is maybe the basis of the recognition, obviously on a strongly perceptive level. During the collective discussion, the student CF justifies his choice in the following way: "The reasoning is… that we took it for granted that the tangent to the parabola passed through that point". Another group (Exp_B) base the spontaneous connection on comparison of the representation on the sheet of paper with that on the calculator screen. In both cases, the iconic representation of the tangent to a curve seems to play a fundamental role.

That strategy, applied by one group (out of six) in Exp_A, can be found in the protocols of three groups (out of five) in Exp_B and two groups (out of six) in Exp_C. However, this causes serious problems in Exp_B and Exp_C in which, because the given function[11] is a polynomial of degree three, the students could not succeed with the algebraic techniques at their disposal. These problems are solved by most of the groups by returning to a local point of view and by adopting one of the two strategies anticipated.

During the collective discussion, the following elements emerge:

- identification of perceptive elements and/or confirmation that the line to be determined is the tangent;
- the relationship between the tangent to the curve and the lines found in its two aspects—tangent line/intersecting lines or tangent line as an approximation of the curve;
- the limits of the numerical procedure;
- the formalisation;
- a new calculation involving orders of magnitude.

In the following analysis, the collective discussion is divided into two parts The first part deals with the first three elements listed above whereas the second part considers the formalisation process and the calculation of the gradient of the tangent line.

In the first part, the teacher deals with the question of the nature of the calculated line, leaving time and space for the problem of connection with the tangent line to emerge. The exchanges between the teacher and students highlight language and gestures that lead to

[11] In the experiments following the first one, the function was changed from a polynomial of degree two to a polynomial of degree three, to make a possible algebraic solution difficult.

the idea of the proximity, firstly spatial and then numerical, of the objects considered. In this part, the teacher prepares the background to pass from a perceptive level to a mathematical level. In other terms, the passage from the work with the artefact to a mathematical text begins.

The exchanges reuse expressions that were shared in the previous session, which therefore serve as a basis for the process being carried out. From Excerpt 5 (below), it emerges that the students resort to the zoom to check the solution that has been found, but that they also use it as a means to improve it (Nos. C28 and C29). The language used is that of approximation: "gets close" (No. C28), "is not exactly a tangent" (No. C28), "close" (No. C35), "even closer" (No. C41), as the gestures (*proximity gestures*) are similar to the proximity of the points to a spatial proximity shown by the fingers (No. C32). In Exp_A, the proximity of the tangent and the curve is conveyed by both the teacher and the students, likewise using the palms of the hands, that represent the geometric objects in question. In this part, a first example of the conversion of this relationship from the linguistic register to the numeric one is noted (Nos. C37 and C41). This conversion, supported by the teacher, then serves as a basis for the subsequent conversion to the symbolic register.

Excerpt 5. Exp_B

After the representation of the line y = $\frac{8}{5}x - \frac{8}{5}$, drawn on the board.

C27. TH: "(…) what is this one like in your opinion?"
C28. TC: "But…not functioning but it gets close because we tried to enlarge it, in the end it is not exactly a tangent but it nearly is."
C29. TH: "Did you find it?"
C30. TC: "Probably with closer points it would have been even more precise."
C31. TH: "So what did you do?"
C32. TC: "We found another point [*the point is indicated with the tips of the fingers*] that belonged to the curve and that was the nearest possible [*small distance between the fingers*] to the point (2; 8/5) and we did it with (9/4; 2)."

The teacher continues with TC's explanation.

C35. TC: "(…) Very close to the point P."
C37. TC: "As the abscissa it could be 2.5 but we put 2.25, that is 9/4."
C38. TH: "And you found this line here, which fits it?"
C39. TC: "It isn't very precise but it is close. If we did it with 2.1, it would be even closer."

The teacher invites TC to explain his idea better.

C41. TC: "If I take an even closer abscissa, like 2.1…"
C42. BF: "But even with 2.0001."
C43. TH: "So if I had used 2.1 or 2.0001 as BF says."
C44. TC: "I think we would get even closer."

Before moving on to the symbolic register, the teacher devotes time to comparison of the curve with the object chosen to approximate it linearly around the chosen point (the tangent), as planned in the a-priori analysis.

Excerpt 6. Exp_B

C100. TH: "Now the question is: can it be calculated exactly [*the tangent at the chosen point*]?"
C101. Ai: "No."

C102. TH: "It can't be calculated exactly?"
C103. TC: "Continuing to get closer… to infinity you can… but…"
C104. TH: "Continuing to get closer to infinity you can… maybe you can or you can't?"
C105. CA: "Between two points I can always find a third and here I will always be able to find a more precise one…I never stop"
C106. TH: "You never stop, so given an approximation I can always find a better one, but I can't find exactly the one on the line…perfect…that one!"

Excerpt 6 shows how the comparison brings into play not only two categories of object at a perceptive level (the "curve" and the "straight line") but also an infinite approximation process, supported by the use of the calculator and the mental iteration of the zooming process carried out in the previous session (No. C103).

Excerpt 7. Exp_B

C165. TH: "So, we went a bit further, ok? So starting from x = 2, the second time we wrote *x* = 2 ? … how much?"
C166. TC: "0.001."
C167. TH: "0.001. Is that ok?", writes *x* = 2 ? 0.001, "Is that better?
C168. BF: "But it would be better if it was even less."
C169. TH: "Even less! So do we actually have to decide how much?"
C170. P: "No."
C171. BF: "Towards zero… that gets close to zero."
C172. P: "2."
C173. TH: "Shall we call it h? Plus a little bit. Is *x* = 2 ? h, ok? What does this little bit have to be like?"
C174. BF: "Small."
C175. TH: "What can it be like? Small or big, but it is better if it is…"
C176. SS: "Very small."
C177. TH: "How small?"
C178. SS: "As small as possible."
C179. BF: "Smaller and smaller."

After having spent time on the question of the approximation and the choice of the points, and on the limits of this procedure for finding the tangent line, the teacher takes the responsibility of moving to the mathematical and symbolic level by writing "2 ? …" (Excerpt 7, No. C165) and introducing '*h*' later (Excerpt 7, No. C173). However, she lets the students explain the nature of the symbol introduced (Excerpt 7, Nos. C171, C178, and C179). The elements that support this interpretation are given by the process carried out up to now. Following the discussion, the expression "similar to 2" appears to refer to the expression "2 + *h*", that seems to express the proximity in another way. The calculation of the gradient of the tangent line is based on these elements, which in this way become operative.

Excerpt 8. Exp_B

C_27. TH: "In reality it is not a line, but it seems to be a line on the calculator. (…) Does anyone want to suggest something more daring? But it's near to zero, isn't it? It's very small and gets mistaken for zero, but if it really gets mistaken, what happens with this function?", (*she indicates* $\frac{h^2+6h+12}{5}$ *which is written on the* *board. Pause*).

C_28. TH: "*h* is near to zero, it gets mistaken for zero, doesn't it? The closer it gets, the better approximation I get. And so if I really reduce it to zero."

C_29. CA: "h^2 becomes insignificant, because it is very small."

C_30. TH: "Eh, h^2 …".

C_31. CA: "Practically…".

C_32. TH: "(…). CA says that h^2 becomes insignificant and therefore remains", (*she writes on the board* $\frac{6h+12}{5}$).

C_35. A1: "Yes, yes."

C_36. A2: "If $h^2$is insignificant, also 6*h* is insignificant".

C_37. A1: "Eh, yes."

C_38. TH: "But is it true? Also 6h?"

C_41. PD: "It remains as twelve fifths."

C_43. A4: "If we take off h twelve fifths remain which is the very normal gradient of a line."

Excerpt 8 shows, in our opinion, how the move to the new calculation of h that involves the limit concept is delicate. The role of the teacher is crucial here: it is necessary to give it a meaning with respect to the algebraic manipulations used for the transformations of the expression, with respect to the previous activities, which suppose h not to be null but as small as desired (Excerpt 5: Nos. C35 and C39), and with respect to the new idea of "negligible objects" that is becoming apparent (No. C_29). This move, which seems to be based on the exploration of the relationship of proximity among the points that emerged previously, draws our attention to the fact that the idea of negligible objects in the symbolic manipulation is expressed in natural language and can effectively support a new calculation, provided that the background has been opportunely prepared beforehand. This fact confirms the relevance of our work with the teachers on the basis of the a-priori analysis.

4.4 Third Session

On the worksheet for the third session, the students are asked to determine the equation for the tangent at a point to the given curves. This involves re-using the calculation that has just been encountered with the orders of magnitude.

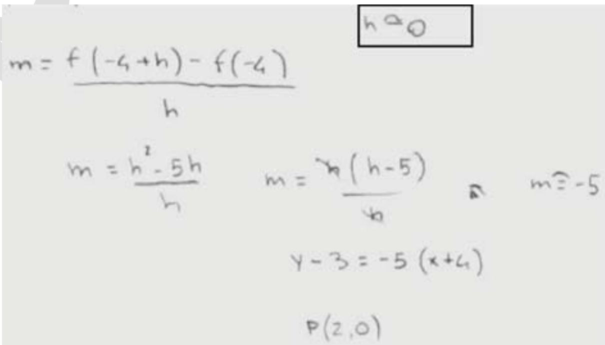


Fig. 7 CA and RM's protocol (Exp_B)

In the students' protocols signs appear (for example, "≈" or "≃" linked to the expression "very small" encountered during the collective discussion) for this new calculation; this also shows some awareness of the magnitude of *h* (for example, Fig. 7).

In the previous parts, we wanted to give an account of some phenomena that occurred during the three experiments with respect to the a-priori analysis that we had done. We shall now move on to the evaluation of the various experiments to check what has previously been constructed.

4.5 Final Tests

The experiments include a test on the notion of the tangent to a curve at a point and/or questionnaires (from which the answers presented at the beginning of the article were taken) for evaluation of the experience gained.

In Exp_A, questions required students to explain the meaning of the term used for microstraightness ("zoomata lineare"), to provide an example and a counter-example and to calculate the equation for the tangent to a curve at a given point. In their answers, the students use different expressions to the question regarding micro-straightness—there are answers using formulations similar to those proposed in class (five students), others that seem to show the influence of the new characterisation of the tangent ("get closer to a line", five students) and other new answers ("degenerate into a line", two students). Almost all the students provide correct examples of functions that satisfy the micro-straightness property. Table 1 below collects elements from the students' answers (the item "from global to local" indicates that the students first consider a curve in the standard window and then an enlargement, as in Fig. 8; the abbreviation "nCr" indicates an incorrect answer).

In those graphical representations one often finds arrows (five students for the "zoomata lineare" and three in the counter-example) used to indicate the zooming process (Fig. 8, where there is also the *micro-straightness sign*) and, in two protocols, there is a square which is probably the trace of the use of the ZoomBox control. In general, the students give priority to the graphic register; this suggests that the concept of micro-straightness has been understood and used above all at an iconic level. Regarding counter-examples, there are less correct answers. In those containing the formula of a function, no point is specified, while in the answers with a formula and standard representation points are given. For the latter, it seems that there may still be interference between the global and local points of view, above all in the symbolic register.

In the calculation of the equation of the tangent line, for which the students are allowed to use the calculator provided they mention it, different strategies appear:

Table 1 Students' answers (Exp_A)

Elements in students' answers	***Example***	***Counter-example***
Formula	3 students	2 students (nCr)
Graphical representation (standard window)	-	2 students
Graphical representation (zoom window)	1 student	1 student
Formula + Graphical representation (standard window)	1 student	4 students (nCr)
Formula + Graphical representation (zoom window)	2 students	1 student
From global to local	3 students	5 students
From global to local + Formula	6 students	-
No answer	-	1 student

– work in the numeric register and approximate solution (the choice of two points): three students;
– the writing of the gradient of the secant line with the introduction of h and calculation: four students;

– the writing of the gradient of the secant line with the introduction of *h* and calculation, using orders of magnitude, with some errors: four students, one of which uses the symbol "≅";
– the writing of the gradient of the secant line with the introduction of *h* and calculation of the limit with the calculator: four students, out of which one solution shows the symbol introduced in class '≈'.

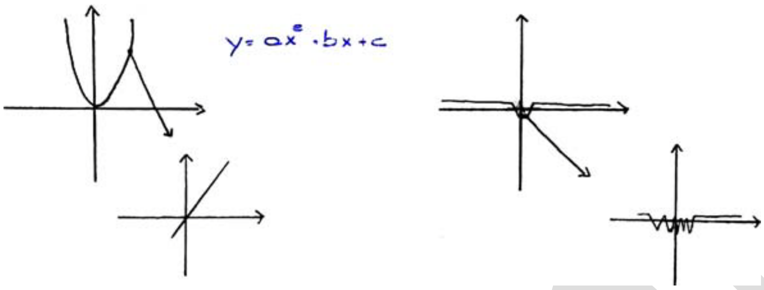


Fig. 8 RE's answer (Exp_A)

The protocol below (Fig. 9) shows a calculation based on orders of magnitude, where the difference from the calculation technique based on just replacing h by zero often mentioned in research, is visible. This confirms the strength of the language introduced in the second session ("*h* is very small" underlined expressions in Fig. 9).

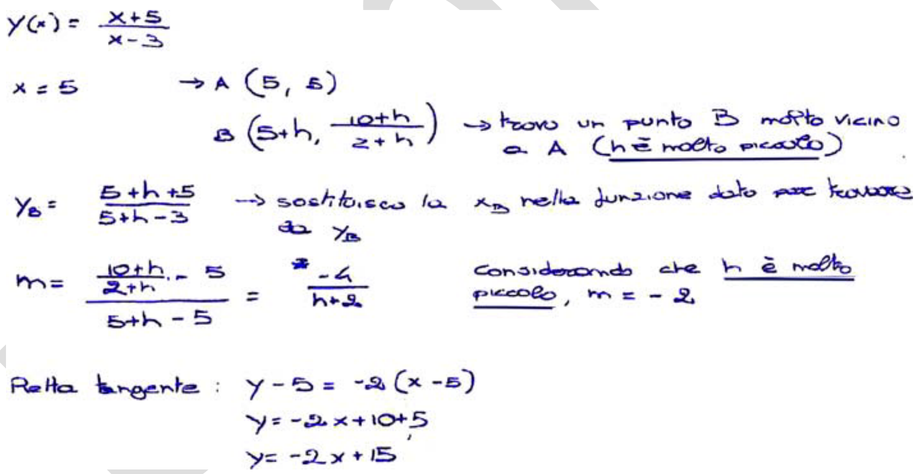
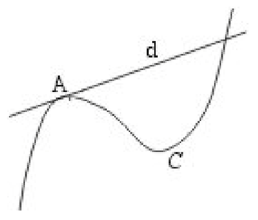
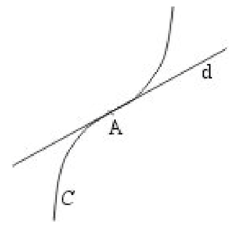
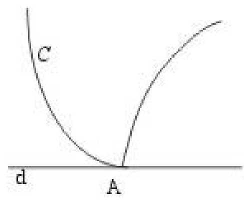


Fig. 9. FS's answer (Exp_A)

In Exp_B, it is decided to give the students the same test on the tangent before (21 students) and after (16 students) the experiment, inspired by the research already quoted, in order to highlight the evolution of the concept of the tangent to a curve and arguments on the existence of the tangent in different cases. The students are asked to indicate if the straight line present in each drawing is tangent to the curve for several configurations. Table 2 shows the data relating to three cases.

Table 2 Comparison of students’ answers (Exp_B)

1		3		4	
					
YES	**NO**	**YES**	**NO**	**YES**	**NO**
Pre: 7 Post: 12	Pre: 14 Post: 4	Pre: 1 Post: 11	Pre: 14 Post: 3	Pre: 17 Post: 1	Pre: 4 Post: 15

Comparison of the answers reveals evident evolution of the relationship with the tangent line, visible in the move from the answer NO to the answer YES from the pre-test to the post-test for the cases in which the tangent intersects the curve (Table 2, case 1). Besides, in the justifications for the answers produced in the post-test there is reference to micro-straightness. In these justifications, the term “zoomata lineare” appears to be associated with a checking process for the existence of the tangent line rather than with a property of the curve.

The test on the tangent is also re-proposed at the end of the Exp_C (21 students), but presenting just four cases, for which the students are requested to indicate whether it is possible to trace the tangent at the points given and, if yes, to do it. Most of the students refer to micro-straightness (for instance, underlined word in Fig. 10) as a criterion to answer the question asked, providing answers that can be compared with those of Exp_B.

Only a few students maintain a global view, providing the argument of the unique intersection between the tangent and the curve. As in the answers to the test in Exp_A, the answers show variations on the theme: “it is micro-straight” (eight students), “it becomes micro-straight” (six students) and “consider the micro-straightness” (one student).

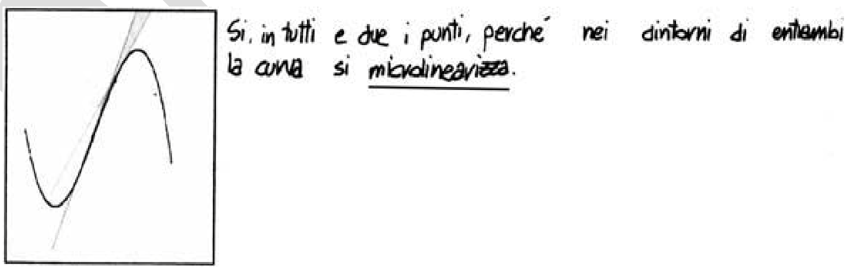


Fig. 10 SM’s answer (Exp_C)

5 Conclusion

This article has presented some characteristic elements of an approach to Calculus in secondary schools by means of the global/local game in a graphic calculator environment. With respect to the research carried out previously in didactics of Calculus that uses graphic manipulations to introduce the first concepts of this field of mathematics, our research places greater emphasis on the problem of the mathematization process of the result of these manipulations and presents a didactic engineering in which this process is given an essential role.

On the basis of cognitive research that draws attention to the role of metaphors and gestures in mathematical conceptualisation, a hypothesis was formed which states that the transformations of the graphical representation of functions, using the zoom-controls and the experience of the perceptive phenomenon that these transformations establish, can bring about the formulation of specific language and the creation of gestures and specific signs which can be used in the processes of constructing mathematical meanings linked to the phenomenon identified. The analyses of the various experiments on which our research is based seem to confirm this hypothesis. The tasks proposed to the students and their management by the teacher allow the construction of the mathematical meaning of a strongly perceptive phenomenon, micro-straightness. The analyses show the role played in this process by the expression and discussion of metaphors and by the sharing of interpretations and gestures linked to the manipulations carried out on the graphical representations. These elements then enable the introduction of a calculation based on orders of magnitude and on the idea of negligible objects that proves to be consistent with the metaphors that emerge. The data collected and their analysis tend to confirm that the idea of micro-straightness, supported by the specific language and gestures associated with it in every class and by the new symbolic computation, become a thinking tool for the students. This can be observed during the sessions, but also in the answers to the tests, that show its evolution. The coined term sums up an experience that can be carried out mentally with new objects, and allows them to be absorbed and interpreted in a certain way. The role as mediator played by the calculator with respect to the new meaning of the tangent line clearly emerges from the analyses of the three experiments.

Even if this article focuses on students' learning and the influence on it of the characteristics of the proposed situations, in this conclusion we would like to insist on the fundamental role played by the teachers in the planning and successful management of the sessions. During the group-work, they are present but in the background, in accordance with the *a-didactic* nature of this phase and they actively participate in the collective discussions, helping class communication. During the latter, they manage the delicate moments of connection between the perceptive and mathematical levels, guiding and supporting the construction of mathematical meaning for micro-straightness and institutionalising the calculation based on the orders of magnitude. The evolution of the engineering design along the three experimentations then shows which elements can be left up to the students and what must be taken charge of by the teacher in the process of the construction of mathematical meanings. The process carried out can be interpreted in terms of a theorem of existence for a learning process initiating the global/local game jointly with calculations based on orders of magnitude in the introduction of Calculus in secondary education. However, our experimental work only explores the beginning of what is necessarily a long-term process; the continuation of the research should take into consideration long-term experiments in order to be able to go beyond this first stage.

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